

ON HEREDITARY COREFLECTIVE SUBCATEGORIES OF **Top**

MARTIN SLEZIAK

ABSTRACT. Let  $A$  be a topological space which is not finitely generated and  $\text{CH}(A)$  denote the coreflective hull of  $A$  in **Top**. We construct a generator of the coreflective subcategory  $\text{SCH}(A)$  consisting of all subspaces of spaces from  $\text{CH}(A)$  which is a prime space and has the same cardinality as  $A$ . We also show that if **A** and **B** are coreflective subcategories of **Top** such that the hereditary coreflective kernel of each of them is the subcategory **FG** of all finitely generated spaces, then the hereditary coreflective kernel of their join  $\text{CH}(\mathbf{A} \cup \mathbf{B})$  is again **FG**.

Keywords: coreflective subcategory, hereditary coreflective subcategory, hereditary coreflective hull, hereditary coreflective kernel, prime space

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## INTRODUCTION

Let  $X$  be a topological space which is not finitely generated and  $\text{SCH}(X)$  be the hereditary coreflective hull of  $X$  in the category **Top** of topological spaces. The aim of this paper is to construct a prime space  $Y_X$  with the same cardinality as  $X$  such that  $\text{SCH}(X) = \text{CH}(Y_X)$  where  $\text{CH}(Y_X)$  is the coreflective hull of  $Y_X$ . Obviously, if  $X$  is finitely generated, then  $\text{CH}(X) = \text{SCH}(X)$ . If  $X$  is not finitely generated, then, using the prime factors of  $X$  we can easily construct a prime space  $P_X$  such that  $\text{SCH}(X) = \text{SCH}(P_X)$ . Thus, it suffices to restrict our investigation to the case of prime spaces.

For the prime space  $C(\omega_0)$  consisting of a convergent sequence and its limit point the problem was studied in [5], where a countable generator for the category  $\text{SCH}(C(\omega_0))$  of subsequential spaces was produced.

Our procedure of constructing a generator  $Y_A$  of the category  $\text{SCH}(A)$  (where  $A$  is a prime space that is not finitely generated) consists of two main steps. In the first step, using similar methods as in [5], we produce a set of special prime spaces which generates  $\text{SCH}(A)$ . Then, in the second step, we construct the generator  $Y_A$  of  $\text{SCH}(A)$  with the required properties.

This construction was inspired by the space  $S_\omega$  from [2] and in the case  $A = C(\omega_0)$  it gives a countable generator for the category of subsequential spaces different from that one presented in [5].

Finally, as an application of some above mentioned results we prove that if **A** and **B** are coreflective subcategories of **Top** such that the hereditary coreflective kernel of **A** as well as the hereditary coreflective kernel of **B** is the category **FG** of finitely generated spaces, then **FG** is also the hereditary coreflective kernel of their join  $\text{CH}(\mathbf{A} \cup \mathbf{B})$ . As a consequence of this result and some results of [9] we obtain that the collection of all those coreflective subcategories of **Top** the hereditary coreflective kernel of which is **FG** and the hereditary coreflective hull of which is

**Top** is closed under the formation of non-empty finite joins (in the lattice of all coreflective subcategories of **Top**) and arbitrary non-empty intersections.

## 1. PRELIMINARIES

We recall some known facts about coreflective subcategories of the category **Top** of topological spaces (see [6]). All subcategories are supposed to be full and isomorphism-closed. The topological sum is denoted by  $\sqcup$ .

Let **A** be a subcategory of **Top**. **A** is *coreflective* if and only if it is closed under the formation of topological sums and quotient spaces. If **A** is a subcategory of **Top** or a class of topological spaces, then the *coreflective hull* of **A** is the smallest coreflective subcategory of **Top** which contains **A** and we denote it by  $\text{CH}(\mathbf{A})$ .  $\text{CH}(\mathbf{A})$  consists of all quotients of topological sums of spaces that belong to **A**. If  $\mathbf{B} = \text{CH}(\mathbf{A})$ , then we say that **A** *generates* **B** and the members of **A** are called *generators* of **B**. If  $\mathbf{B} = \text{CH}(\{X\})$ , then **B** is called *simple generated* and  $X$  is said to be a *generator* of **B**. We use the notation  $\mathbf{B} = \text{CH}(X)$  in this case.

Let **A** be a subcategory of **Top** and let **SA** denote the subcategory of **Top** consisting of all subspaces of spaces from **A**. Then the following result is known (see [8, Remark 2.4.4(5)] or [3, Proposition 3.1]).

**Proposition 1.1.** *If **A** is a coreflective subcategory of **Top**, then **SA** is also a coreflective subcategory of **Top**. (**SA** is the hereditary coreflective hull of **A**.)*

By **FG** we denote the category of all finitely generated spaces. It is well known (see e.g. [6]) that if  $X$  is not finitely generated, then  $\mathbf{FG} \subseteq \text{CH}(X)$ .

We say that a subcategory **A** of **Top** is *hereditary* if with each topological space  $X$  it contains also all its subspaces. It is well known that the category of all finitely generated spaces and all its subcategories that are coreflective in **Top** are hereditary.

Some known hereditary coreflective subcategories of **Top** are  $\mathbf{Gen}(\alpha)$  and  $\mathbf{Top}(\alpha)$ , where  $\alpha$  is an infinite cardinal.  $\mathbf{Gen}(\alpha)$  is the subcategory of all spaces having tightness not exceeding  $\alpha$ .  $\mathbf{Top}(\alpha)$  is the category of all topological spaces such that the intersection of every family of open sets, which has cardinality less than  $\alpha$ , is an open set.

Let  $A$  be a topological space. We say that  $A$  is a *prime space* if it has precisely one accumulation point. The following assertion is obvious.

**Lemma 1.2.** *Let  $X$  be a prime space with an accumulation point  $a$  and let  $Y$  be a subspace of  $X$  containing the point  $a$ , then the map  $f: X \rightarrow Y$ , defined by  $f(x) = x$  for  $x \in Y$  and  $f(x) = a$  for  $x \in X \setminus Y$ , is a quotient map.*

Given a topological space  $X$  and a point  $a \in X$ , denote by  $X_a$  the space constructed by making each point, other than  $a$ , isolated with  $a$  retaining its original neighborhoods. (I.e. a subset  $U \subseteq X$  is open in  $X_a$  if and only if  $a \notin U$  or there exists an open subset  $V$  of  $X$  such that  $a \in V \subseteq U$ .) The topological space  $X_a$  is called *prime factor of  $X$  at the point  $a$* . It is clear that any prime factor is either a prime space or a discrete space.

**Proposition 1.3** ([3, Proposition 3.5]). *If **A** is a hereditary coreflective subcategory of **Top** with  $\mathbf{FG} \subseteq \mathbf{A}$ , then for each  $X \in \mathbf{A}$  and each  $a \in X$  the prime factor  $X_a$  of  $X$  at  $a$  belongs to **A**.*

Let  $A$  be a prime space with an accumulation point  $a$ . A subspace  $B$  of  $A$  is said to be a *prime subspace* of  $A$  if  $B$  is a prime space (i.e.  $a \in B$  and  $\overline{B \setminus \{a\}} \ni a$ ).

**Lemma 1.4.** *Let  $(A_i; i \in I)$  be a family of prime spaces and let  $a_i \in A_i$  be an accumulation point of  $A_i$  for  $i \in I$ . A topological space  $X$  belongs to  $\text{CH}(\{A_i; i \in I\})$  if and only if for every non-closed subset  $M$  of  $X$  there exists  $i \in I$ , a prime subspace  $B$  of  $A_i$  and a continuous map  $f: B \rightarrow X$  such that  $f[B \setminus \{a_i\}] \subseteq M$  and  $f(a_i) \notin M$ .*

*Proof.* Let  $\mathbf{B} \subseteq \mathbf{Top}$  be the class of all topological spaces satisfying the given condition. First we show that  $\mathbf{B}$  is a coreflective subcategory of  $\mathbf{Top}$ . It is evident that  $\mathbf{B}$  is closed under the formation of topological sums. Now let  $X \in \mathbf{B}$  and  $q: X \rightarrow Y$  be a quotient map. Let  $M$  be a non-closed subset of  $Y$ . Then  $q^{-1}[M]$  is a non-closed subset of  $X$ ,  $X \in \mathbf{B}$ , so that there exists  $i \in I$ , a prime subspace  $B$  of  $A_i$  and a continuous map  $g: B \rightarrow X$  such that  $g[B \setminus \{a_i\}] \subseteq q^{-1}[M]$  and  $g(a_i) \notin q^{-1}[M]$ . Then for  $f = q \circ g: B \rightarrow Y$  we get  $f[B \setminus \{a_i\}] \subseteq M$  and  $f(a_i) \notin M$ . Hence,  $Y \in \mathbf{B}$  and  $\mathbf{B}$  is a coreflective subcategory of  $\mathbf{Top}$ .

Since evidently  $A_i \in \mathbf{B}$  for each  $i \in I$ , we have  $\text{CH}(\{A_i; i \in I\}) \subseteq \mathbf{B}$ . To prove the reverse inclusion we construct a quotient map from a sum of subspaces of  $A_i$  to arbitrary space  $X \in \mathbf{B}$ . (Every subspace of  $A_i$  belongs to  $\text{CH}(A_i)$  by Lemma 1.2.)

Let  $X \in \mathbf{B}$ . Let  $f_j: B_j \rightarrow X$ ,  $j \in J$ , be the family of all continuous maps such that  $B_j$  is a prime subspace of some  $A_i$ ,  $i \in I$ . Let  $D(X)$  be the discrete space on the set  $X$  and  $\text{id}_X: D(X) \rightarrow X$  be the identity map. It is easy to check that the map  $f: D(X) \sqcup (\coprod_{j \in J} B_j) \rightarrow X$  given by the maps  $\text{id}_X$  and  $f_j, j \in J$ , is a quotient map.  $\square$

Cardinals are initial ordinals where each ordinal is the (well-ordered) set of its predecessors. We denote the class of all ordinals by  $\text{ON}$ . If  $\alpha$  is a cardinal, then by  $\alpha^+$  we denote the cardinal which is a successor of  $\alpha$ . A net in a topological space defined on an ordinal  $\alpha$  we call an  $\alpha$ -sequence.

From now on we assume that  $A$  is a prime space with an accumulation point  $a$  which is not finitely generated and the tightness of the space  $A$  is  $t(A) = \alpha$ .

## 2. CLOSURE OPERATOR DESCRIBING $\text{CH}(A)$

The notion of sequential closure was used in [5] when studying sequential and subsequential spaces. Now we introduce a corresponding closure operator for the subcategory  $\text{CH}(A)$ .

Let  $X$  be an arbitrary space and  $M \subseteq X$ . The set  $M_1 = \{x \in X : \text{there exists a prime subspace } B \text{ of } A \text{ and a continuous map } f: B \rightarrow X \text{ such that } f[B \setminus \{a\}] \subseteq M \text{ and } f(a) = x\}$  is called the *A-closure* of  $M$ . Using transfinite induction we can define the set  $M_\beta$  (the  $\beta$ -th *A-closure* of  $M$ ) for each ordinal  $\beta$  as follows.  $M_0 = M$ ,  $M_{\beta+1} = (M_\beta)_1$  for each ordinal  $\beta$  and  $M_\gamma = \bigcup_{\beta < \gamma} M_\beta$  for each limit ordinal  $\gamma > 0$ . Put  $\widetilde{M} = \bigcup_{\beta \in \text{ON}} M_\beta$ .

Evidently  $(\widetilde{M})_1 = \widetilde{M}$ ,  $\widetilde{M} \subseteq \overline{M}$ . It is also clear that  $M_\beta \subseteq M_\gamma$  holds for  $\beta < \gamma$ . If  $A \subseteq B \subseteq X$ , then  $A_\beta \subseteq B_\beta$  for each ordinal  $\beta$  and  $\widetilde{A} \subseteq \widetilde{B}$ . If  $M_\beta = M_{\beta+1}$  for some ordinal  $\beta$ , then  $\widetilde{M} = M_\beta$ .

The following proposition characterizes the spaces belonging to  $\text{CH}(A)$  using the closure operator  $M \mapsto \widetilde{M}$ . It is a special case of [8, Theorem 3.1.7] which includes more general cases of closure operators.

**Proposition 2.1.** *A topological space  $X$  belongs to  $\text{CH}(A)$  if and only if  $\overline{M} = \widetilde{M}$  for every subset  $M \subseteq X$ .*

*Proof.* Let  $X \in \text{CH}(A)$  and  $M \subseteq X$ . Then  $(\widetilde{M})_1 \setminus \widetilde{M} = \emptyset$ , so that by Lemma 1.4  $\widetilde{M}$  is closed and  $\widetilde{M} = \overline{M}$ .

Conversely, if  $\overline{M} = \widetilde{M}$  for each  $M \subseteq X$  and  $M$  is non-closed, then  $M_1 \setminus M \neq \emptyset$  and there exists a prime subspace  $B$  of  $A$  and a continuous map  $f: B \rightarrow X$  such that  $f[B \setminus \{a\}] \subseteq M$  and  $f(a) \notin M$ . Hence, according to Lemma 1.4, we conclude that  $X \in \text{CH}(A)$ .  $\square$

**Proposition 2.2.** *Let  $A$  be a prime space with an accumulation point  $a$ ,  $X \in \text{CH}(A)$  and  $\alpha = t(A)$ . Then for every subset  $M \subseteq X$  it holds  $M_{\alpha^+} = \overline{M}$ .*

*Proof.* It suffices to prove that  $(M_{\alpha^+})_1 = M_{\alpha^+}$ . Let  $c \in (M_{\alpha^+})_1$ . Then there exists a prime subspace  $B$  of  $A$  and a continuous map  $f: B \rightarrow X$  with  $f(a) = c$  and  $f[B \setminus \{a\}] \subseteq M_{\alpha^+}$ . Since  $t(A) = \alpha$  and  $a \in \overline{B \setminus \{a\}}$ , there exists  $C \subseteq B \setminus \{a\}$  with  $\text{card } C \leq \alpha$  such that  $a \in \overline{C}$ . The subspace  $B_1 = C \cup \{a\}$  of  $A$  is a prime subspace,  $f|_{B_1}: B_1 \rightarrow X$  is continuous and  $f|_{B_1}[C] \subseteq M_{\alpha^+}$ .

For each  $x \in C$  choose  $\beta_x < \alpha^+$  such that  $x \in M_{\beta_x}$  ( $\alpha^+$  is a limit ordinal). Since  $\text{card } C \leq \alpha < \alpha^+$  and  $\alpha^+$  is a regular cardinal we obtain that  $\gamma = \sup\{\beta_x, x \in C\} < \alpha^+$ . Then  $C \subseteq M_\gamma$  and, obviously,  $f|_{B_1}(a) = f(a) = c \in M_{\gamma+1} \subseteq M_{\alpha^+}$ . Thus,  $(M_{\alpha^+})_1 \subseteq M_{\alpha^+}$ .  $\square$

### 3. $A$ -SUM

The notion of  $A$ -sum is a special case of the brush defined in [8] and a generalization of the sequential sum introduced in [2]. The sequential sum was used in [5] for constructing the set of “canonical” prime spaces which generates the category of subsequential spaces. The notion of the  $A$ -sum will be used in a similar way to produce the set of special prime spaces that generates  $\text{SCH}(A)$ .

**Definition 3.1.** Let  $A$  be a prime space with an accumulation point  $a \in A$ . Let us denote  $B := A \setminus \{a\}$ . Let for each  $b \in B$   $X_b$  be a topological space and  $x_b \in X_b$ . Then the  $A$ -sum  $\sum_A \langle X_b, x_b \rangle$  is the topological space on the set  $F = A \cup (\bigcup_{b \in B} \{b\} \times (X_b \setminus \{x_b\}))$  such that the map  $\varphi: A \sqcup (\coprod_{b \in B} X_b) \rightarrow F$  given by  $\varphi(x) = x$  for  $x \in A$ ,  $\varphi(x) = (b, x)$  for  $x \in X_b \setminus \{x_b\}$  and  $\varphi(x_b) = b$  for every  $b \in B$  is a quotient map. (We assume  $A$  and all  $\{b\} \times X_b$  to be disjoint.) The map  $\varphi$  will be called the *defining map* of the  $A$ -sum.

Often it will be clear from the context what we mean under  $A$  and we will abbreviate the notation of the  $A$ -sum to  $\sum \langle X_b, x_b \rangle$  or  $\sum X_b$ . The  $A$ -sum is obtained simply by identifying every  $x_b \in X_b$  with the point  $b \in A$ . It is easy to see that the subspace  $\varphi[X_b]$  is homeomorphic to  $X_b$  and  $A$  is also a subspace of the  $A$ -sum  $\sum \langle X_b, x_b \rangle$ .

The  $A$ -sum is defined using topological sum and quotient map, thus if  $\mathbf{A}$  is a coreflective subcategory of  $\mathbf{Top}$  and  $\mathbf{A}$  contains  $A$  and all  $X_b$ 's, then the  $A$ -sum  $\sum X_b$  belongs to  $\mathbf{A}$ .

The following lemma follows easily from the definition of the  $A$ -sum.

**Lemma 3.2.** *A subset  $U \subseteq \sum_A \langle X_b, x_b \rangle$  is open (closed) if and only if  $U \cap A$  is open (closed) in  $A$  and  $U \cap \varphi[X_b]$  is open (closed) in  $\varphi[X_b]$  for every  $b \in B$ .*

Let for every  $b \in B$   $X_b$  and  $Y_b$  be topological spaces,  $x_b \in X_b$ ,  $y_b \in Y_b$  and let  $f_b: X_b \rightarrow Y_b$  be a function with  $f_b(x_b) = y_b$ . Then we can define a map  $f =: \sum f_b: \sum_A \langle X_b, x_b \rangle \rightarrow \sum_A \langle Y_b, y_b \rangle$  by  $f(b, x) = (b, f_b(x))$  for  $x \in X_b \setminus \{x_b\}$  and  $f(x) = x$  for  $x \in A$ . Let us note that  $f \circ \varphi_1|_{X_b} = \varphi_2|_{Y_b} \circ f_b$  where  $\varphi_1$  and  $\varphi_2$  are the defining maps of the  $A$ -sums  $\sum X_b$  and  $\sum Y_b$  respectively.

We will need the following simple lemma:

**Lemma 3.3.** *Let  $f: X \rightarrow Y$  be a quotient map,  $A \subseteq Y$  and let  $f$  be one-to-one outside  $A$ . Then  $f|_{f^{-1}[A]}: f^{-1}[A] \rightarrow A$  is a quotient map.*

**Lemma 3.4.** *Let  $A$  be a prime space with an accumulation point  $a$  and  $B = A \setminus \{a\}$ . Let for every  $b \in B$   $f_b: X_b \rightarrow Y_b$  be a map between topological spaces,  $x_b \in X_b$ ,  $y_b \in Y_b$  and  $f_b(x_b) = y_b$ .*

- (i) *If all  $f_b$ 's are continuous, then  $\sum f_b$  is continuous.*
- (ii) *If all  $f_b$ 's are quotient maps, then  $\sum f_b$  is a quotient map.*
- (iii) *If all  $f_b$ 's are embeddings, then  $\sum f_b$  is an embedding.*
- (iv) *If all  $f_b$ 's are homeomorphisms, then  $\sum f_b$  is a homeomorphism.*
- (v) *Let  $C$  be a prime subspace of  $A$ . Then  $\sum_C \langle X_b, x_b \rangle$  is a subspace of the space*

$$\sum_A \langle X_b, x_b \rangle.$$

*Proof.* Put  $f = \sum f_b$  and let  $\varphi_1, \varphi_2$  be the defining maps of the  $A$ -sums  $\sum \langle X_b, x_b \rangle$ ,  $\sum \langle Y_b, y_b \rangle$  respectively. Let us denote  $id_A \sqcup (\prod_{b \in B} f_b)$  by  $h$ . In this situation the following diagram commutes.

$$\begin{array}{ccc} A \sqcup (\prod X_b) & \xrightarrow{h} & A \sqcup (\prod Y_b) \\ \varphi_1 \downarrow & & \varphi_2 \downarrow \\ \sum \langle X_b, x_b \rangle & \xrightarrow{f} & \sum \langle Y_b, y_b \rangle \end{array}$$

The validity of (i) and (ii) follows easily from the fact that  $\varphi_1$  and  $\varphi_2$  are quotient maps.

(iii) Now, suppose that all  $f_b$ 's are embeddings. W.l.o.g. we can assume that  $X_b \subseteq Y_b$  and  $f_b$  is the inclusion of  $X_b$  into  $Y_b$  for every  $b \in B$ . Let  $X'$  be the subspace of the space  $\sum Y_b$  on the set  $\sum X_b$ . We have the following situation:

$$\begin{array}{ccc} A \sqcup (\prod X_b) & \xhookrightarrow{h} & A \sqcup (\prod Y_b) \\ \varphi_1 \downarrow & & \varphi_2 \downarrow \\ X' & \xhookrightarrow{f} & \sum Y_b \end{array}$$

We only need to prove that  $X'$  has the quotient topology with respect to  $\varphi_1$ , because this implies that  $X' = \sum X_b$  and  $f$  is an embedding of  $X' = \sum X_b$  to  $\sum Y_b$ . But  $\varphi_2$  is one-to-one outside the set  $A \sqcup (\prod X_b)$  and Lemma 3.3 implies that  $\varphi_1$  is a quotient map.

(iv) It is an easy consequence of (ii) and (iii). (v) It follows easily from the definition of the  $A$ -sum.  $\square$

**Corollary 3.5.** *Let  $A$  be a prime space with an accumulation point  $a$  and let  $C$  be a prime subspace of  $A$ . Let for every  $b \in A \setminus \{a\}$   $X_b$  be a topological space and*

$x_b \in X_b$ . Let for every  $b \in C$   $Y_b$  be a subspace of  $X_b$  such that  $x_b \in Y_b$ . Then  $\sum_C \langle Y_b, x_b \rangle$  is a subspace of the space  $\sum_A \langle X_b, x_b \rangle$ .

Let us note, that if for every  $b \in A \setminus \{a\}$   $f_b$  is an embedding which maps isolated points of  $X_b$  to isolated points of  $Y_b$ , then the embedding  $\sum f_b$  has the same property.

#### 4. THE SETS $TS_\gamma$ , $TSS_\gamma$

In this section we construct the set of special prime spaces that generates  $SCH(A)$  (where  $A$  is a prime space which is not finitely generated and  $t(A) = \alpha$ ). We start with defining the set  $TS_\gamma$  of topological spaces for each ordinal  $\gamma < \alpha^+$ .

Let  $TS_0 = \emptyset$  and  $TS_1$  be the set of all prime subspaces of  $A$ .

If  $\beta \geq 1$  is an ordinal, then  $TS_{\beta+1}$  consists of all  $B$ -sums  $\sum_B \langle X_b, x_b \rangle$  where  $B$  is a prime subspace of  $A$ , each  $X_b \in TS_\beta$  and  $x_b = a$ .

If  $\gamma > 0$  is a limit ordinal, then  $TS_\gamma = \bigcup_{\beta < \gamma} TS_\beta$ .

Sometimes, if we want to emphasize which prime space  $A$  is used to construct this set, we use the notation  $TS_\gamma(A)$ .

Every space belonging to  $TS_\gamma$  contains  $B$  as a subspace and therefore it contains  $a$ . All spaces from  $TS_\gamma$  are constructed from  $A$  using  $B$ -sums, where  $B \in CH(A)$ , thus  $TS_\gamma \subseteq CH(A)$  for each  $\gamma$ .

The following lemma is a generalization of [5, Lemma 6.2].

**Lemma 4.1.** *Let  $X$  be a topological space and  $M \subseteq X$ . If  $p \in M_\beta \setminus M_\gamma$  for any  $\gamma < \beta$ , then there exists a space  $S \in TS_\beta$  and a continuous map  $f: S \rightarrow X$ , which maps all isolated points of  $S$  into  $M$  and maps only the point  $a$  to  $p$ .*

*Proof.* For  $\beta = 1$  the claim follows from the definition of  $M_1$ .

From the definition of  $M_\beta$  it follows that  $\beta$  is a non-limit ordinal. According to Proposition 2.2  $\beta < \alpha^+$ . Suppose the assertion is true for any subset  $K$  of  $X$  and for any  $\beta' < \beta$ .

For a non-limit  $\beta > 1$  there exists a prime subspace  $B$  of  $A$  and a continuous map  $f: B \rightarrow X$  such that  $f(a) = p$  and  $f[B \setminus \{a\}] \subseteq M_{\beta-1}$ .

If  $\beta - 1$  is non-limit, we can moreover assume that  $f[B \setminus \{a\}] \subseteq M_{\beta-1} \setminus M_{\beta-2}$ . (If necessary, we choose  $B' = \{b \in B : f(b) \in M_{\beta-1} \setminus M_{\beta-2}\}$  and  $f' = f|_{B'}$ .  $B'$  is a prime subspace of  $A$ , otherwise we get  $x \in M_{\beta-1}$ .)

If  $\beta - 1$  is a limit ordinal, then for each point  $x \in M_{\beta-1}$  there exists the smallest ordinal  $\gamma < \beta - 1$  such that  $x \in M_\gamma$ . Obviously,  $\gamma$  is a non-limit ordinal.

Thus for each  $x \in f[B \setminus \{a\}]$  there exists a continuous map  $f_x: S_x \rightarrow X$ , where  $S_x \in TS_{\beta-1}$ , which sends all isolated points of  $S_x$  into  $M$  and  $a$  to  $x$ .

Then  $\sum_B \langle S_{f(b)}, a \rangle \in TS_\beta$  and we can define a map  $g: \sum_B \langle S_{f(b)}, a \rangle \rightarrow X$  such that  $g|_B = f$  and  $g|_{\{x\} \times (S_x \setminus \{a\})}(x, y) = f_x(y)$  for  $y \in S_x \setminus \{a\}$ . Clearly,  $g$  maps isolated points into  $M$ . It remains only to show that  $g$  is continuous.

The defining map  $\varphi: B \sqcup (\prod_{b \in B \setminus \{a\}} S_{f(b)}) \rightarrow \sum_B \langle S_{f(b)}, a \rangle$  is a quotient map. Thus,  $g: \sum_B \langle S_{f(b)}, a \rangle \rightarrow X$  is continuous if and only if  $g \circ \varphi$  is continuous. But  $g \circ \varphi|_B = f$  and  $g \circ \varphi|_{S_x} = f_x$  are continuous, thus  $g$  is continuous.  $\square$

For any  $S \in TS_\gamma$  we denote by  $P(S)$  the subspace of the space  $S$  which consists of all isolated points of  $S$  and of the point  $a$ . Clearly,  $P(S)$  is a prime space. We

denote by  $TSS_\gamma$  the set of all spaces  $P(S)$  where  $S \in TS_\gamma$ . The above lemma implies:

**Lemma 4.2.** *If  $p \in M_\beta$  and  $p \notin M_\gamma$  for any  $\gamma < \beta$ , then there exists a space  $T \in TSS_\beta$  and a continuous map  $f: T \rightarrow X$ , which maps all isolated points of the space  $T$  into  $M$  and such that  $f(a) = p$ .*

**Proposition 4.3.** *SCH(A) is generated by the set  $\bigcup_{\gamma < \alpha^+} TSS_\gamma$ .*

*Proof.* Let  $X \in \text{SCH}(A)$ . According to Lemma 1.4 it suffices to prove that for any subset  $M \subseteq X$  and any  $x \in \overline{M} \setminus M$  there exists  $T \in \bigcup_{\gamma < \alpha^+} TSS_\gamma$  and a continuous map  $f: T \rightarrow X$  such that  $f(a) = x$  and  $f[T \setminus \{a\}] \subseteq M$ .

Since  $X \in \text{SCH}(A)$  there exists  $Y \in \text{CH}(A)$  such that  $X$  is a subspace of  $Y$ . Denote by  $\overline{M}^Y$  the closure of  $M$  in  $Y$ . Then  $\overline{M} = \overline{M}^Y \cap X$  and  $x \in \overline{M}^Y \setminus M$  in  $Y$ . By Proposition 2.2  $\overline{M}^Y = M_{\alpha^+} = \bigcup_{\beta < \alpha^+} M_\beta$ . Let  $\beta$  be the smallest ordinal with  $x \in M_\beta$ . Then  $\beta > 0$  and for any  $\gamma < \beta$   $x \notin M_\gamma$ . By Lemma 4.1 there exists  $S \in TS_\beta$  and a continuous map  $f: S \rightarrow Y$  with  $f(a) = x$  and  $f(c) \in M$  for any isolated point of  $S$ . Then  $P(S) \in TSS_\gamma$  and  $f[P(S)] \subseteq X$ . Hence,  $f|_{P(S)}: P(S) \rightarrow X$  is a continuous map satisfying the required conditions. Consequently,  $X \in \text{CH}(\bigcup_{\gamma < \alpha^+} TSS_\gamma)$ .  $\square$

**Remark 4.4.** It can be easily seen that if we define the sets  $T'S_\gamma$ ,  $\gamma < \alpha^+$ , similarly as the sets  $TS_\gamma$  but we use only the  $A$ -sums (and not all  $B$ -sums for prime subspaces  $B$  of  $A$ ) and then we put  $T'SS_\gamma = \{P(S) : S \in T'S_\gamma\}$  we obtain the set  $\bigcup_{\gamma < \alpha^+} T'SS_\gamma$  which also generates  $\text{SCH}(A)$ . This follows from the fact that any space from  $\bigcup_{\gamma < \alpha^+} TSS_\gamma$  is a prime subspace of some space from  $\bigcup_{\gamma < \alpha^+} T'SS_\gamma$ .

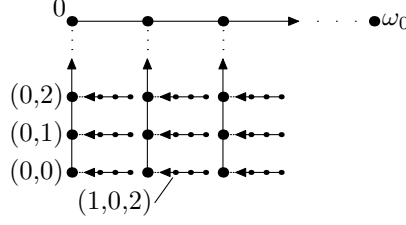
Similarly, if we put  $T'SS'_\gamma = \{S_a : S \in T'S_\gamma\}$  ( $S_a$  is the prime factor of  $S$  at  $a$ ), then the set  $\bigcup_{\gamma < \alpha^+} T'SS'_\gamma$  generates  $\text{SCH}(A)$  because  $\bigcup_{\gamma < \alpha^+} T'SS'_\gamma \subseteq \text{SCH}(A)$  and for every  $S \in \bigcup_{\gamma < \alpha^+} T'S_\gamma$   $P(S)$  is a subspace of  $S_a$ .

## 5. THE SPACES $A_\omega$ AND $(A_\omega)_a$

The space  $A_\omega$  is defined similarly as  $S_\omega$  in [2] using the  $A$ -sum and the space  $A$  instead of the sequential sum and the space  $C(\omega_0)$ . We start with defining the space  $A_n$  for each  $n \in \mathbb{N}$  putting  $A_1 = A$  and  $A_{n+1} = \sum_A \langle A_n, a \rangle$ . Clearly,  $A_1$  is a subspace of  $A_2$  and if  $A_{n-1}$  is a subspace of  $A_n$ , then, according to Lemma 3.4,  $A_n = \sum_A \langle A_{n-1}, a \rangle$  is a subspace of  $A_{n+1} = \sum_A \langle A_n, a \rangle$ . Hence,  $A_n$  is a subspace of  $A_{n+1}$  for each  $n \in \mathbb{N}$ .

The Figure 1 represents the space  $A_3$  for  $A = C(\omega_0)$ . (The space  $C(\omega_0)$  is defined in Example 5.7.)

The space  $A_\omega$  is a topological space defined on the set  $\bigcup_{n \in \mathbb{N}} A_n$  such that a subset  $U$  of  $\bigcup_{n \in \mathbb{N}} A_n$  is open in  $A_\omega$  if and only if  $U \cap A_n$  is open in  $A_n$  for every  $n \in \mathbb{N}$ . It is obvious that for every  $n \in \mathbb{N}$  the space  $A_n$  is a subspace of  $A_\omega$  and  $A_\omega$  is a quotient space of the topological sum  $\coprod_{n \in \mathbb{N}} A_n$ . Consequently,  $A_\omega$  belongs to

FIGURE 1. The space  $A_3$  for  $A = C(\omega_0)$ 

$\text{CH}(A)$ . Observe that  $A_\omega$  can be considered as an inductive limit of its subspaces  $A_n$ ,  $n \in \mathbb{N}$ .

Similarly as the space  $S_\omega$  in [2] the space  $A_\omega$  has the following important property.

**Proposition 5.1.**  $A_\omega = \sum_A \langle A_\omega, a \rangle$

*Proof.* Put  $X = \sum_A \langle A_\omega, a \rangle$ . For each  $n \in A$  the space  $A_n$  is a subspace of  $A_\omega$  and it follows that  $A_{n+1} = \sum_A \langle A_n, a \rangle$  is a subspace of  $X$  (Lemma 3.4). Obviously,  $A = A_1$  is also a subspace of  $X$  and we obtain that for each  $n \in \mathbb{N}$   $A_n$  is a subspace of  $X$ . Clearly,  $X = \bigcup_{n \in \mathbb{N}} A_n$ . To finish the proof it suffices to check that if  $U$  is a subset of  $X$  and  $U \cap A_n$  is open in  $A_n$  for each  $n \in \mathbb{N}$ , then  $U$  is open in  $X$ .

Let us denote by  $A_n^b$  the subspace of  $X$  on the set  $\{b\} \cup (\{b\} \times (A_n \setminus \{a\}))$  and by  $A_\omega^b$  the subspace of  $X$  on the set  $\{b\} \cup (\{b\} \times (A_\omega \setminus \{a\}))$ . Clearly,  $A_n^b$  is homeomorphic to  $A_n$  and  $A_\omega^b$  is homeomorphic to  $A_\omega$ ,  $A_n^b$  is a subspace of  $A_\omega^b$  and a subset  $V$  of  $A_\omega^b$  is open in  $A_\omega^b$  if and only if  $V \cap A_n^b$  is open in  $A_n^b$  for each  $n \in \mathbb{N}$ .

If  $U \subseteq X$  and  $U \cap A_n$  is open in  $A_n$  for all  $n \in \mathbb{N}$ , then  $U \cap A$  is open in  $A$  and  $U \cap A_{n+1}$  is open in  $A_{n+1} = \sum_A \langle A_n, a \rangle$  for all  $n \in \mathbb{N}$ . Then  $U \cap A_n^b$  is open in  $A_n^b$  for each  $n \in \mathbb{N}$  and  $b \in A \setminus \{a\}$  and it follows that  $U \cap A_\omega^b$  is open in  $A_\omega^b$  for each  $b \in A \setminus \{a\}$ . Hence,  $U$  is open in  $X$ .  $\square$

The following lemma is evident.

**Lemma 5.2.**  $\text{card } A_\omega = \text{card } A$

**Lemma 5.3.** For every ordinal  $\gamma$ ,  $1 \leq \gamma < \alpha^+$  and every space  $S \in TS_\gamma$  the space  $S$  is a subspace of  $A_\omega$ . (Clearly, the point  $a$  of  $S$  coincides with the point  $a$  of  $A_\omega$ .)

*Proof.* If  $\gamma = 1$ , then  $S = B$  is a prime subspace of  $A = A_1$ . Let  $\gamma$  be an ordinal,  $1 < \gamma < \alpha^+$  and the assertion hold for every ordinal  $\beta$ ,  $1 \leq \beta < \gamma$ . If  $S = \sum_B X_b \in TS_\gamma$ , then for each  $b \in B \setminus \{a\}$   $X_b \in TS_{\beta_b}$  with  $1 \leq \beta_b < \gamma$ . Hence, for each  $b \in B \setminus \{a\}$ ,  $X_b$  is a subspace of  $A_\omega$  and, according to Corollary 3.5,  $S$  is a subspace of  $A_\omega = \sum_A \langle A_\omega, a \rangle$ .  $\square$

**Theorem 5.4.** Let  $(A_\omega)_a$  be the prime factor of the space  $A_\omega$  at  $a$ . Then  $(A_\omega)_a$  is a prime space,  $\text{CH}((A_\omega)_a) = \text{SCH}(A)$  and  $\text{card}(A_\omega)_a = \text{card } A$ .



*Proof.* Evidently,  $(A_\omega)_a$  is a prime space and  $\text{card}(A_\omega)_a = \text{card } A$ . Since  $A_\omega$  belongs to  $\text{CH}(A) \subseteq \text{SCH}(A)$ , according to Proposition 1.3  $(A_\omega)_a$  belongs to  $\text{SCH}(A)$ . Hence, it suffices to check that  $\bigcup_{\gamma < \alpha^+} TSS_\gamma \subseteq \text{CH}((A_\omega)_a)$ .

Let  $T \in \bigcup_{\gamma < \alpha^+} TSS_\gamma$ . Then there exists an ordinal  $\gamma$ ,  $1 \leq \gamma < \alpha^+$ , and  $S \in TSS_\gamma$  such that  $T = P(S)$ . By Lemma 5.3  $S$  is a subspace of  $A_\omega$  and, clearly, it follows that  $T = P(S)$  is a subspace of  $(A_\omega)_a$ . Consequently, there exists a quotient map  $(A_\omega)_a \rightarrow T$  and we obtain that  $T$  belongs to  $\text{CH}((A_\omega)_a)$ .  $\square$

Finally, let  $X$  be an arbitrary topological space which is not finitely generated and  $\{X_c, c \in Y\}$  be the set of all prime factors of  $X$  that are not discrete spaces. Denote by  $A_X$  the quotient space of the topological sum  $\coprod_{c \in Y} (\{c\} \times X_c)$  obtained by collapsing all points of the subset  $\{(c, c), c \in Y\}$  of the space  $\coprod_{c \in Y} (\{c\} \times X_c)$  into one point  $a$ .

The space  $A_X$  is a prime space which is not finitely generated,  $a$  is the accumulation point of  $A_X$ ,  $\text{card } A_X = \text{card } X$  and the following statement holds:

**Theorem 5.5.**  $\text{SCH}(X) = \text{CH}(((A_X)_\omega)_a)$ ,  $((A_X)_\omega)_a$  is a prime space and  $\text{card}((A_X)_\omega)_a = \text{card } X$ .

*Proof.* Evidently,  $A_X \in \text{SCH}(X)$ ,  $X \in \text{CH}(A_X)$  and therefore  $\text{SCH}(X) = \text{SCH}(A_X)$ . According to Theorem 5.4  $\text{SCH}(A_X) = \text{CH}(((A_X)_\omega)_a)$ ,  $((A_X)_\omega)_a$  is a prime space and  $\text{card}((A_X)_\omega)_a = \text{card } A_X = \text{card } X$ .  $\square$

Recall, that a topological space  $X$  belongs to **Top**( $\omega_1$ ) if and only if every countable intersection of open subsets of  $X$  is open in  $X$  and **Top**( $\omega_1$ ) is a hereditary coreflective subcategory of **Top**. If the space  $A$  belongs to **Top**( $\omega_1$ ), then  $\text{SCH}(A) \subseteq \text{Top}(\omega_1)$  and we can find smaller (and simpler) set of generators of  $\text{SCH}(A)$  than the set  $\bigcup_{\gamma < \alpha^+} TSS_\gamma(A)$  constructed in Proposition 4.3.

**Proposition 5.6.** If  $A \in \text{Top}(\omega_1)$ , then  $\text{SCH}(A) = \text{CH}(\{P(A_n); 0 < n < \omega_0\})$ .

*Proof.* It suffices to show that  $(A_\omega)_a \in \text{CH}(\{P(A_n); n < \omega_0\})$ . Clearly, each  $P(A_n)$  is a subspace of  $(A_\omega)_a$ . Denote by  $i_n: P(A_n) \hookrightarrow (A_\omega)_a$  the corresponding embedding and by  $f: \coprod_{n \in \mathbb{N}} P(A_n) \rightarrow (A_\omega)_a$  the continuous map given by the maps  $i_n$ ,  $n \in \mathbb{N}$ . It is easy to see that this map is surjective. We claim that  $f$  is also a quotient map.

It suffices to show that if  $a \in U \subseteq (A_\omega)_a$  and  $U \cap P(A_n)$  is open in  $P(A_n)$  for each  $n$ ,  $0 < n < \omega_0$ , then  $U$  is open in  $(A_\omega)_a$ . Since  $P(A_n)$  is a subspace of  $A_\omega$ , there exists an open subset  $W_n$  of  $A_\omega$  such that  $W_n \cap P(A_n) = U \cap P(A_n)$ . Put  $W = \bigcap_{0 < n < \omega_0} W_n$ . The set  $W$  is open in  $A_\omega$  since  $A_\omega$  belongs to **Top**( $\omega_1$ ). We have  $W \cap P(A_n) \subseteq U \cap P(A_n)$  and  $\bigcup_{0 < n < \omega_0} P(A_n) = A_\omega$ , therefore  $W \subseteq U$ . Obviously,  $a \in W$ . Hence, the set  $U$  is open in  $(A_\omega)_a$ .  $\square$

Next we present some special cases of our construction.

**Example 5.7. Sequential spaces.** Recall that subspaces of sequential spaces are called subsequential. The category **Seq** of sequential spaces is the coreflective hull of the space  $C(\omega_0)$ . The space  $C(\omega_0)$  is the topological space on the set  $\omega_0 + 1 = \omega_0 \cup \{\omega_0\}$  such that all points of  $\omega_0$  are isolated and a set containing  $\omega_0$  is

open if and only if its complement is finite. (Equivalently, the topology of  $C(\omega_0)$  is the order topology given by the usual well-ordering of  $\omega_0 + 1$ .) The space  $C(\omega_0)_\omega$  is homeomorphic to  $S_\omega$  defined in [2]. Our results imply that the prime factor of the space  $C(\omega_0)_\omega$  at  $\omega_0$  is a generator of the category of subsequential spaces. Another countable generator of this category was constructed before in [5].

**Example 5.8.** *The coreflective hull of the space  $C(\alpha)$ .* Let  $\alpha$  be a regular cardinal and  $C(\alpha)$  be the topological space on the set  $\alpha + 1 = \alpha \cup \{\alpha\}$  such that all points of  $\alpha$  are isolated and a set containing  $\alpha$  is open if and only if its complement has cardinality less than  $\alpha$ . It is well known that  $X$  belongs to  $\text{CH}(C(\alpha))$  if and only if a subset  $V \subseteq X$  is closed in  $X$  whenever for each  $\alpha$ -sequence of points from  $V$  the set  $V$  contains also all limits of this  $\alpha$ -sequence. The subcategories  $\text{SCH}(C(\alpha))$  are minimal elements of the collection of all hereditary coreflective subcategories of **Top** above **FG**. We use the subcategories  $\text{SCH}(C(\alpha))$  in the next section. Our construction yields the generator  $(C(\alpha)_\omega)_\alpha$  of  $\text{SCH}(C(\alpha))$  which has cardinality  $\alpha$ .

## 6. SUBCATEGORIES OF **Top** HAVING **FG** AS THEIR HEREDITARY COREFLECTIVE KERNEL

Recall that a *hereditary coreflective kernel* of a subcategory **A** of **Top** is the largest hereditary coreflective subcategory of **Top** contained in **A**. We denote it by  $\text{HCK}(\mathbf{A})$ . In this section we prove that if **A** and **B** are coreflective subcategories of **Top** such that  $\text{HCK}(\mathbf{A}) = \text{HCK}(\mathbf{B}) = \mathbf{FG}$ , then also  $\text{HCK}(\text{CH}(\mathbf{A} \cup \mathbf{B})) = \mathbf{FG}$ . The analogous result does not hold for infinite countable joins of coreflective subcategories. This problem is closely related to the subcategories  $\text{SCH}(C(\alpha))$  because (see [3, Theorem 4.8]) **FG** is the hereditary coreflective kernel of a coreflective subcategory **A** of **Top** if and only if  $\mathbf{FG} \subseteq \mathbf{A}$  and for any regular cardinal  $\alpha$  the category  $\text{SCH}(C(\alpha))$  is not contained in **A**.

In [7, Problem 7] H. Herrlich and M. Hušek suggest to study coreflective subcategories of **Top** such that their hereditary coreflective hull is the whole category **Top** (i.e.  $\mathbf{SA} = \mathbf{Top}$ ) and their hereditary coreflective kernel is the subcategory **FG**. In the paper [9] it is shown that there exists the smallest such subcategory of **Top** and the collection of all such subcategories of **Top** is closed under the formation of arbitrary non-empty intersections. In this section we prove that this collection is also closed under the formation of non-empty finite joins without being closed under the formation of infinite countable joins in the lattice of all coreflective subcategories of **Top**.

Throughout this section we will apply the results obtained in preceding sections to prime spaces  $C(\alpha)$ ,  $\alpha$  being a regular cardinal, defined in Example 5.8. Note that  $\alpha$  is an accumulation point of  $C(\alpha)$  and  $t(C(\alpha)) = \alpha$  for any regular cardinal  $\alpha$ . Since any prime subspace of  $C(\alpha)$  is homeomorphic to  $C(\alpha)$  it suffices to use only  $C(\alpha)$ -sums in the definition of  $TSS_\gamma$ . For instance, if  $n$  is a natural number, then  $TSS_n$  as well as  $TSS_n$  contain precisely one space.

In order to prove the main result of this section, we first prove that if  $\text{SCH}(C(\alpha)) \subseteq \text{CH}(\mathbf{A} \cup \mathbf{B})$  for some coreflective subcategories **A**, **B** of **Top**, then one of these subcategories contains  $\text{SCH}(C(\alpha))$ . We show it separately for the case  $\alpha = \omega_0$  and  $\alpha \geq \omega_1$ .

We start with the case  $\alpha = \omega_0$  where we can use some results presented in the paper [5]. As the sets  $TSS_\gamma$ ,  $\gamma < \omega_1$ , introduced in [5] do not coincide with the sets  $TSS_\gamma(C(\omega_0))$  defined in Section 4 we denote the sets used in [5] by  $TSS'_\gamma(C(\omega_0))$ .

The next lemma follows from [5, Theorem 7.1], resp. [5, Corollary 7.2].

**Lemma 6.1.** *The category  $\mathbf{SSeq} = \text{SCH}(C(\omega_0))$  of subsequential spaces is the coreflective hull of the set  $\bigcup_{\gamma < \omega_1} TSS'_\gamma(C(\omega_0))$ .*

As a consequence of [5, Theorem 7.1] and [5, Theorem 6.4] we obtain:

**Lemma 6.2.** *If  $\beta < \gamma < \omega_1$ , then  $TSS'_\beta(C(\omega_0)) \subseteq \text{CH}(TSS'_\gamma(C(\omega_0)))$ .*

The following result concludes the part of this section concerning the subcategory  $\text{SCH}(C(\omega_0))$ .

**Proposition 6.3.** *If  $\text{SCH}(C(\omega_0)) \subseteq \text{CH}(\bigcup_{i \in I} \mathbf{A}_i)$ ,  $\mathbf{A}_i$  is a coreflective subcategory of **Top** for every  $i \in I$  and  $\text{card } I \leq \omega_0$ , then there exists  $i_0 \in I$  such that  $\text{SCH}(C(\omega_0)) \subseteq \mathbf{A}_{i_0}$ .*

*Proof.* Put  $\beta_i = \sup\{\beta : TSS'_\beta(C(\omega_0)) \subseteq \mathbf{A}_i\}$  for  $i \in I$ . Since  $\sup \beta_i = \omega_1$  (Lemma 6.1) and  $\omega_1$  is a regular cardinal, there exists  $i_0 \in I$  such that  $\beta_{i_0} = \omega_1$ . By Lemma 6.1 and Lemma 6.2 we get that the coreflective subcategory  $\mathbf{A}_{i_0}$  contains the subcategory  $\mathbf{SSeq} = \text{SCH}(C(\omega_0))$ .  $\square$

Next we want to prove a result analogous to Proposition 6.3 for the space  $C(\alpha)$ , where  $\alpha \geq \omega_1$  is a regular cardinal. In the case  $\alpha \geq \omega_1$  the desired result holds only for non-empty finite joins of coreflective subcategories of **Top**.

Recall that  $C(\alpha)_1 = C(\alpha)$  and  $C(\alpha)_{n+1} = \sum_{C(\alpha)} \langle C(\alpha)_n, \alpha \rangle$ . According to Corollary 3.5 we obtain that  $P(C(\alpha)_{n+1}) = P(\sum_{C(\alpha)} \langle P(C(\alpha)_n), \alpha \rangle)$  and it is easy to see

that  $\alpha^{n+1} \cup \{\alpha\}$  is the underlying set of the space  $P(C(\alpha)_{n+1})$  and the subspace of  $\sum P(C(\alpha)_n)$  on the set  $\{\eta\} \cup (\{\eta\} \times \alpha^n)$  is homeomorphic to  $P(C(\alpha)_n)$  for each  $\eta < \alpha$ . To simplify the notation we will write  $C(\alpha)_n^-$  instead of  $P(C(\alpha)_n)$ .

The following result is a special case of Proposition 5.6.

**Proposition 6.4.** *If  $\alpha \geq \omega_1$  is a regular cardinal, then  $\text{SCH}(C(\alpha)) = \text{CH}(\{C(\alpha)_n^-; 0 < n < \omega_0\})$ .*

**Lemma 6.5.** *Let  $\alpha \geq \omega_1$  be a regular cardinal. If  $M$  is a subset of  $C(\alpha)_n$  such that  $\alpha \in \overline{M}$  and  $M$  contains only isolated points of  $C(\alpha)_n$ , then there exists a subset  $M' \subseteq M$  such that the subspace of the space  $C(\alpha)_n$  on the set  $\overline{M'}$  is homeomorphic to  $C(\alpha)_n$ .*

*Proof.* The case  $n = 1$  is clear. Let the assertion be true for  $m$ . Denote the subspace of  $C(\alpha)_{m+1} = \sum_{C(\alpha)} C(\alpha)_m$  on the set  $\{\eta\} \cup (\{\eta\} \times (C(\alpha)_m \setminus \{\alpha\}))$ , where  $\eta < \alpha$ , by  $C(\alpha)_m^\eta$ .

Put  $B = \overline{M} \cap C(\alpha)$ . Then  $B$  is a prime subspace of  $C(\alpha)$ , for each  $\eta \in B \setminus \{\alpha\}$  all points of the set  $M_\eta = M \cap C(\alpha)_m^\eta$  are isolated in the space  $C(\alpha)_m^\eta$  and  $\eta \in \overline{M_\eta}$  in  $C(\alpha)_m^\eta$  (observe that  $\overline{M_\eta}$  in  $C(\alpha)_m^\eta$  coincides with  $\overline{M_\eta}$  in  $C(\alpha)_{m+1}$  because  $C(\alpha)_m^\eta$  is closed in  $C(\alpha)_{m+1}$ ). Since  $C(\alpha)_m^\eta$  is homeomorphic to  $C(\alpha)_m$  by the induction assumption we obtain that there exists a subset  $M'_\eta \subseteq M_\eta$  such that  $\eta \in \overline{M'_\eta}$  and the subspace  $\overline{M'_\eta}$  of  $C(\alpha)_m^\eta$  is homeomorphic to some space  $C(\alpha)_m$ .

Let  $B' = B \setminus \{\alpha\}$  and  $M' = \bigcup_{\eta \in B'} M'_\eta$ . Clearly,  $M' \subseteq M$ ,  $\overline{M'} = \bigcup_{\eta \in B'} \overline{M'_\eta} \cup \{\alpha\}$  in  $S$  and  $\overline{M'_\eta}$  is homeomorphic to  $C(\alpha)_m$  for each  $\eta \in B'$ .

The subspace  $B$  of  $C(\alpha)$  is homeomorphic to  $C(\alpha)$  and it is easy to check that  $\overline{M'}$  is homeomorphic to  $\sum_{C(\alpha)} C(\alpha)_m = C(\alpha)_{m+1}$ .  $\square$

**Corollary 6.6.** *Let  $\alpha \geq \omega_1$  be a regular cardinal,  $0 < n < \omega_0$ . Then every prime subspace  $T$  of  $C(\alpha)_n^-$  is homeomorphic to  $C(\alpha)_n^-$ .*

*Proof.* Put  $M = T \setminus \{\alpha\}$ . Clearly,  $\alpha \in \overline{M}$ . According to Lemma 6.5 there exists a subset  $M'$  of  $M$  such that the subspace  $M' \cup \{\alpha\}$  of  $C(\alpha)_n^-$  is homeomorphic to  $C(\alpha)_n^-$ . It follows from the proof of Lemma 6.5 that  $M \setminus M'$  is a discrete clopen subspace of  $C(\alpha)_n^-$  with cardinality at most  $\alpha$ . Hence,  $T = M \cup \{\alpha\}$  is homeomorphic to  $C(\alpha)_n^-$  as well.  $\square$

**Proposition 6.7.** *Let  $\alpha \geq \omega_1$  be a regular cardinal and  $0 < n < \omega_0$ . If  $C(\alpha)_n^- \in \text{CH}(\bigcup_{i \in I} \mathbf{A}_i)$ , where all  $\mathbf{A}_i$ 's are coreflective subcategories of **Top**, then there exists  $i_0 \in I$  such that  $C(\alpha)_n^- \in \mathbf{A}_{i_0}$ .*

*Proof.* The space  $C(\alpha)_n^-$  is a prime space with an accumulation point  $\alpha$ . If  $C(\alpha)_n^- \in \text{CH}(\bigcup_{i \in I} \mathbf{A}_i)$ , then there exists a quotient map  $f: \prod_{i \in I} B_i \rightarrow C(\alpha)_n^-$ , where  $B_i$  belongs to  $\mathbf{A}_i$  for each  $i \in I$ . Put  $f_i = f|_{B_i}$  and let  $A_i$  be the space on the set  $f_i[B_i]$  endowed with the quotient topology with respect to  $f_i$  for each  $i \in I$ .

The topology of every space  $A_i$  is finer than the topology of the corresponding subspace of  $C(\alpha)_n^-$  and it follows that  $A_i$  is either discrete or prime space. Clearly, a set  $U \subseteq C(\alpha)_n^-$  is open in  $C(\alpha)_n^-$  if and only if  $U \cap A_i$  is open in  $A_i$  for each  $i \in I$  and  $A_i \in \mathbf{A}_i$ . Obviously, there exists  $i_0 \in I$  such that  $\alpha$  is an accumulation point of  $A_{i_0}$  (otherwise  $\alpha$  would be isolated in  $C(\alpha)_n^-$ ).

We show that  $C(\alpha)_n^- \in \text{CH}(A_{i_0})$ . Let  $M$  be a non-closed subset of  $C(\alpha)_n^-$ . It suffices to find a continuous map  $f: A_{i_0} \rightarrow C(\alpha)_n^-$  such that  $f[A_{i_0} \setminus \{\alpha\}] \subseteq M$  and  $f(\alpha) = \alpha$ . According to Corollary 6.6 the subspace on the set  $M \cup \{\alpha\}$  is homeomorphic to  $C(\alpha)_n^-$ . Let us denote the homeomorphism from  $C(\alpha)_n^-$  to  $M \cup \{\alpha\}$  by  $g$ . Moreover, there is a continuous map  $i: A_{i_0} \rightarrow C(\alpha)_n^-$  defined by  $i(x) = x$  for each  $x \in A_{i_0}$ . The desired continuous map is  $f = g \circ i$ .  $\square$

If  $X$  and  $Y$  are prime spaces, then a continuous map  $f: X \rightarrow Y$  is called a *prime map* if it maps only the accumulation point of  $X$  to the accumulation point of  $Y$ .

**Lemma 6.8.** *Let  $\alpha \geq \omega_1$  be a regular cardinal and  $0 < m < n < \omega_0$ . There exists a quotient prime map  $g: C(\alpha)_n^- \rightarrow C(\alpha)_m^-$ .*

*Proof.* Obviously, it suffices to prove the lemma for  $n = m + 1$ . In this case  $C(\alpha)_{m+1}^- = P(\sum C(\alpha)_m^-)$  is a topological space on the set  $\{\alpha\} \cup \alpha^{m+1}$  and  $C(\alpha)_m^-$  is a topological space on the set  $\{\alpha\} \cup \alpha^m$ . We define a map  $g: C(\alpha)_{m+1}^- \rightarrow C(\alpha)_m^-$  by  $g(\alpha) = \alpha$  and  $g((\eta, x)) = x$  for all  $(\eta, x) \in C(\alpha)_{m+1}^- \setminus \{\alpha\}$ . It is easy to check that the map  $g$  is continuous and quotient.  $\square$

**Corollary 6.9.** *If  $\alpha \geq \omega_1$  is a regular cardinal and  $0 < m < n < \omega_0$ , then  $C(\alpha)_m^- \in \text{CH}(C(\alpha)_n^-)$ .*

**Proposition 6.10.** *If  $\alpha$  is a regular cardinal and  $\text{SCH}(C(\alpha)) \subseteq \text{CH}(\mathbf{A} \cup \mathbf{B})$ , then  $\text{SCH}(C(\alpha)) \subseteq \text{CH}(\mathbf{A})$  or  $\text{SCH}(C(\alpha)) \subseteq \text{CH}(\mathbf{B})$ .*

*Proof.* Since the case  $\alpha = \omega_0$  follows immediately from Proposition 6.3 we can assume that  $\alpha \geq \omega_1$ .

By Proposition 6.7 for each  $n$ ,  $0 < n < \omega_0$ , the space  $C(\alpha)_n^-$  belongs either to **A** or to **B**. By Lemma 6.8 we have a quotient map  $f: C(\alpha)_n^- \rightarrow C(\alpha)_m^-$  for each  $n > m$ . Hence, one of these two coreflective categories contains all spaces  $C(\alpha)_n^-$  and, consequently, it contains  $\text{SCH}(C(\alpha))$ .  $\square$

Now we can state the main result of this section.

**Theorem 6.11.** *If **A**, **B** are coreflective subcategories of the category **Top** and  $\text{HCK}(\mathbf{A}) = \text{HCK}(\mathbf{B}) = \mathbf{FG}$ , then  $\text{HCK}(\text{CH}(\mathbf{A} \cup \mathbf{B})) = \mathbf{FG}$ .*

*Proof.* Suppose the contrary. Then according to [3, Theorem 4.8] there exists a regular cardinal  $\alpha$  with  $\text{SCH}(C(\alpha)) \subseteq \text{CH}(\mathbf{A} \cup \mathbf{B})$ . Proposition 6.10 implies that  $\text{SCH}(C(\alpha)) \subseteq \mathbf{A}$  or  $\text{SCH}(C(\alpha)) \subseteq \mathbf{B}$ , contradicting the assumption that the hereditary coreflective kernel of both these categories is **FG**.  $\square$

Let  $\mathcal{C}$  be the conglomerate of all coreflective subcategories of **Top**. It is well known that  $\mathcal{C}$  partially ordered by  $\subseteq$  is a complete lattice. Denote by  $\mathcal{K}$  the conglomerate of all coreflective subcategories **A** of **Top** with  $\text{HCK}(\mathbf{A}) = \mathbf{FG}$ . The above theorem says that  $\mathcal{K}$  is closed under the formation of non-empty finite joins in  $\mathcal{C}$ . We next show that  $\mathcal{K}$  fails to be closed under the formation of infinite countable joins in  $\mathcal{C}$ . In the concrete we prove that if  $\alpha \geq \omega_1$  is a regular cardinal, then all categories  $\text{CH}(C(\alpha)_n^-)$  belong to  $\mathcal{K}$ . According to Proposition 6.4 the category  $\text{SCH}(C(\alpha))$  is the join of this family in  $\mathcal{C}$  and, evidently,  $\text{SCH}(C(\alpha)) \notin \mathcal{K}$ . The proof is divided into three auxiliary lemmas.

**Lemma 6.12.** *Let  $\alpha \geq \omega_1$  be a regular cardinal and  $2 \leq n < \omega_0$ . If there exists a prime map  $f: C(\alpha)_n^- \rightarrow C(\alpha)_{n+1}^-$ , then there exists a prime map  $f': C(\alpha)_n^- \rightarrow C(\alpha)_{n+1}^-$  such that  $f'[\{\xi\} \times \alpha^{n-1}] \cap (\bigcup_{\eta < \xi} \{\eta\} \times \alpha^n) = \emptyset$  for each  $\xi < \alpha$ .*

*Proof.* Let  $f: C(\alpha)_n^- \rightarrow C(\alpha)_{n+1}^-$  be a prime map. Denote by  $B_\xi$  the subspace of  $\sum C(\alpha)_{n-1}^-$  on the set  $\{\xi\} \cup (\{\xi\} \times \alpha^{n-1})$  where  $\xi < \alpha$ . The subspace  $B_\xi$  is homeomorphic to  $C(\alpha)_{n-1}^-$ .

For each  $\xi < \alpha$  the set  $f^{-1}[\{\alpha\} \cup (\bigcup_{\eta \geq \xi} \{\eta\} \times \alpha^n)]$  is open in  $C(\alpha)_n^-$ , therefore there exists an ordinal  $\gamma < \alpha$  such that for each  $\gamma' > \gamma$  the set  $\{\gamma'\} \cup (f^{-1}[\bigcup_{\eta \geq \xi} \{\eta\} \times \alpha^n] \cap B_{\gamma'})$  is open in  $B_{\gamma'}$ . Hence, we can define an increasing sequence  $(\gamma_\xi)_{\xi < \alpha}$  such that  $C_\xi := \{\gamma_\xi\} \cup (f^{-1}[\bigcup_{\eta \geq \xi} \{\eta\} \times \alpha^n] \cap B_{\gamma_\xi})$  is open in  $B_{\gamma_\xi}$ . Clearly,  $f[C_\xi \setminus \{\gamma_\xi\}] \subseteq \bigcup_{\eta \geq \xi} \{\eta\} \times \alpha^n$ .

According to Corollary 6.6 the subspace of  $B_{\gamma_\xi}$  on the set  $C_\xi$  is homeomorphic to  $C(\alpha)_{n-1}^-$ . Hence, for each  $\xi < \alpha$  we can define an embedding  $h_\xi: C(\alpha)_{n-1}^- \hookrightarrow \sum C(\alpha)_{n-1}^-$  such that  $h_\xi[C(\alpha)_{n-1}^-] = C_\xi$ . It is easy to see that the map  $h: \sum C(\alpha)_{n-1}^- \rightarrow \sum C(\alpha)_{n-1}^-$  given by  $h(\xi) = \gamma_\xi$  for each  $\xi < \alpha$ ,  $h(\alpha) = \alpha$  and  $h(\xi, x) = h_\xi(x)$  for each  $\xi < \alpha$  and  $x \in \alpha^{n-1}$  is also an embedding. Put  $A_\xi = \{\xi\} \times \alpha^{n-1}$  ( $A_\xi \subseteq B_\xi$ ). Then  $h[A_\xi] \subseteq h_\xi[C(\alpha)_{n-1}^-] = C_\xi$  and  $f[h[A_\xi]] \subseteq f[C_\xi \setminus \{\gamma_\xi\}] \subseteq \bigcup_{\eta \geq \xi} \{\eta\} \times \alpha^n$ . Consequently,  $f \circ h[A_\xi] \cap (\bigcup_{\eta < \xi} \{\eta\} \times \alpha^n) = \emptyset$  and the prime map  $f' = f \circ (h|_{C(\alpha)_n^-}): C(\alpha)_n^- \rightarrow C(\alpha)_{n+1}^-$  is a prime map satisfying the required condition.  $\square$

**Lemma 6.13.** *Let  $\alpha \geq \omega_1$  be a regular cardinal and  $0 < n < \omega_0$ . Then there exists no prime map from  $C(\alpha)_n^-$  to  $C(\alpha)_{n+1}^-$ .*

*Proof.* First let  $n = 1$ . For each  $\gamma < \alpha$  the set  $\{\gamma\} \times \alpha$  is closed in  $C(\alpha)_2^-$ . Consequently,  $f^{-1}[\{\gamma\} \times \alpha]$  is closed in  $C(\alpha)$ , hence it contains less than  $\alpha$  points and there exists a set  $U_\gamma \subseteq \alpha$  with  $\text{card}(\alpha \setminus U_\gamma) < \alpha$  such that  $(\{\gamma\} \times U_\gamma) \cap f[C(\alpha)] = \emptyset$ . Thus,  $W = \{\alpha\} \cup \left(\bigcup_{\gamma < \alpha} \{\gamma\} \times U_\gamma\right)$  is an open neighborhood of  $\alpha$  in  $C(\alpha)_2^-$  such that  $f^{-1}[W] = \{\alpha\}$  and this contradicts the continuity of  $f$ .

Let  $n > 1$  and the lemma hold for  $n - 1$ . Suppose that there exists a prime map  $f: C(\alpha)_n^- \rightarrow C(\alpha)_{n+1}^-$ . By Lemma 6.12 we can assume w.l.o.g. that  $f[\{\xi\} \times \alpha^{n-1}] \cap (\bigcup_{\eta < \xi} \{\eta\} \times \alpha^n) = \emptyset$  for each  $\xi < \alpha$ .

Recall the definition of the quotient prime map  $g: C(\alpha)_n^- \rightarrow C(\alpha)_{n-1}^-$  from Lemma 6.8. The map  $g$  is defined by  $g(\alpha) = \alpha$  and  $g(\eta, x) = x$  for  $\eta < \alpha$ ,  $x \in \alpha^{n-1}$ .

Put  $A_\xi = \{\xi\} \times \alpha^{n-1}$ . Let us denote the subspace of  $\sum C(\alpha)_{n-1}^-$  on the set  $\{\xi\} \cup A_\xi = \{\xi\} \cup (\{\xi\} \times \alpha^{n-1})$  by  $B_\xi$  for each  $\xi < \alpha$ . Clearly,  $B_\xi$  is homeomorphic to  $C(\alpha)_{n-1}^-$ . We define a map  $f_\xi: B_\xi \rightarrow C(\alpha)_{n+1}^-$  by  $f_\xi(\xi) = \alpha$  and  $f_\xi(\xi, x) = f(\xi, x)$  for each  $x \in \alpha^{n-1}$ .

The map  $g \circ f_\xi: B_\xi \rightarrow C(\alpha)_n^-$  cannot be continuous, otherwise we get a prime map from a space homeomorphic to  $C(\alpha)_{n-1}^-$  to the space  $C(\alpha)_n^-$ . Therefore there exists an open subset of  $C(\alpha)_n^-$  such that inverse image of this set is not open in  $B_\xi$ . This set can be written in the form  $U_\xi \cup \{\alpha\}$ , where  $\alpha \notin U_\xi$ , and we get that the set

$$f_\xi^{-1}[g^{-1}[U_\xi \cup \{\alpha\}]] = f_\xi^{-1}[\{\alpha\} \cup (\bigcup_{\eta < \alpha} (\{\eta\} \times U_\xi))] = \{\xi\} \cup (B_\xi \cap f^{-1}[\bigcup_{\eta < \alpha} \{\eta\} \times U_\xi])$$

is not open in  $B_\xi$ .

Put  $V_\xi = \bigcap_{\eta \leq \xi} U_\eta$  for  $\xi < \alpha$ . The family  $V_\xi$  is non-increasing and it has the same properties as the family  $U_\xi$ . Each  $V_\xi \cup \{\alpha\}$  is open in  $C(\alpha)_n^-$ , because  $C(\alpha)_n^-$  belongs to  $\mathbf{Top}(\alpha)$  ( $\text{SCH}(C(\alpha)) \subseteq \mathbf{Top}(\alpha)$ ). The set  $\{\xi\} \cup (B_\xi \cap f^{-1}[\bigcup_{\eta < \alpha} \{\eta\} \times V_\xi])$  is not open in  $B_\xi$  since  $B_\xi$  is a prime space with an accumulation point  $\xi$  (and  $\{\xi\} \cup (B_\xi \cap f^{-1}[\bigcup_{\eta < \alpha} \{\eta\} \times V_\xi]) \subseteq \{\xi\} \cup (B_\xi \cap f^{-1}[\bigcup_{\eta < \alpha} \{\eta\} \times U_\xi])$ ).

Finally let us put  $W = \bigcup_{\xi < \alpha} \{\xi\} \times V_\xi$ . The set  $W \cup \{\alpha\}$  is open in  $C(\alpha)_{n+1}^-$ . We claim that  $f^{-1}[\{\alpha\} \cup W]$  is not open in  $C(\alpha)_n^-$ . It suffices to show that  $\{\xi\} \cup (f^{-1}[\{\alpha\} \cup W] \cap B_\xi)$  is not open in  $B_\xi$  for each  $\xi < \alpha$ .

Clearly,  $B_\xi = A_\xi \cup \{\xi\}$  and we get  $\{\xi\} \cup (f^{-1}[\{\alpha\} \cup W] \cap B_\xi) = \{\xi\} \cup (f^{-1}[\bigcup_{\eta < \alpha} \{\eta\} \times V_\eta] \cap A_\xi)$ . We have  $f[A_\xi] \cap (\bigcup_{\eta < \xi} \{\eta\} \times \alpha^{n-1}) = \emptyset$ , hence  $f^{-1}[\bigcup_{\eta < \alpha} \{\eta\} \times V_\eta] \cap A_\xi = f^{-1}[\bigcup_{\eta \geq \xi} \{\eta\} \times V_\eta] \cap A_\xi$  and we obtain

$$\begin{aligned} \{\xi\} \cup (f^{-1}[\{\alpha\} \cup W] \cap B_\xi) &= \{\xi\} \cup (f^{-1}[\bigcup_{\eta \geq \xi} \{\eta\} \times V_\eta] \cap B_\xi) \subseteq \\ &\subseteq \{\xi\} \cup (f^{-1}[\bigcup_{\eta \geq \xi} \{\eta\} \times V_\xi] \cap B_\xi) \subseteq \{\xi\} \cup (f^{-1}[\bigcup_{\eta < \alpha} \{\eta\} \times V_\xi] \cap B_\xi). \end{aligned}$$

The latter set is not open in  $B_\xi$  therefore  $\{\xi\} \cup (f^{-1}[\{\alpha\} \cup W] \cap B_\xi)$  is not open in  $B_\xi$  as well.  $\square$

**Lemma 6.14.** *Let  $\alpha \geq \omega_1$  be a regular cardinal and  $0 < n < \omega_0$ . Then  $\text{HCK}(\text{CH}(C(\alpha)_n^-)) = \mathbf{FG}$ .*

*Proof.* Recall (see [6]) that if  $\gamma > \delta$ , then  $\mathbf{Top}(\gamma) \cap \mathbf{Gen}(\delta) = \mathbf{FG}$ . For  $\beta < \alpha$  we have  $\text{SCH}(C(\beta)) \subseteq \mathbf{Gen}(\beta)$  and  $C(\alpha)_n^- \in \mathbf{Top}(\alpha)$ , hence  $\text{SCH}(C(\beta)) \not\subseteq$

$\text{CH}(C(\alpha)_n^-)$ . Similarly if  $\beta > \alpha$ , then  $\text{SCH}(C(\beta)) \subseteq \mathbf{Top}(\beta)$  and  $C(\alpha)_n^- \in \mathbf{Gen}(\alpha)$ . Thus,  $\text{SCH}(C(\beta)) \not\subseteq \text{CH}(C(\alpha)_n^-)$ .

By Lemma 6.13 and Lemma 1.4  $C(\alpha)_{n+1}^- \notin \text{CH}(C(\alpha)_n^-)$  (every prime subspace of  $C(\alpha)_n^-$  is homeomorphic to  $C(\alpha)_n^-$  and  $C(\alpha)_{n+1}^- \in \text{SCH}(C(\alpha))$ , therefore  $\text{SCH}(C(\alpha)) \not\subseteq \text{CH}(C(\alpha)_{n+1}^-)$  as well.  $\square$

Denote by  $\mathcal{L}$  the collection of all coreflective subcategories **A** of **Top** such that  $\mathbf{SA} = \mathbf{Top}$  and  $\text{HCK}(\mathbf{A}) = \mathbf{FG}$ . In the paper [9] it is shown that  $\mathcal{L}$  has the smallest element  $\mathbf{A}_0 = \text{CH}(\{S^\alpha; \alpha \text{ is a cardinal}\})$ , where  $S$  is the Sierpiński doubleton, and  $\mathcal{L}$  is closed under the formation of arbitrary non-empty intersections. This together with Theorem 6.11 yields:

**Theorem 6.15.** *The collection  $\mathcal{L}$  is closed under the formation of non-empty intersections, non-empty finite joins in  $\mathcal{C}$  and has the smallest element.*

**Proposition 6.16.** *There is no maximal coreflective subcategory **A** of **Top** such that  $\text{HCK}(\mathbf{A}) = \mathbf{FG}$ . Consequently, the collection  $\mathcal{L}$  has no maximal element.*

*Proof.* Suppose that **A** is maximal coreflective subcategory of **Top** with the property  $\text{HCK}(\mathbf{A}) = \mathbf{FG}$ . Let  $\alpha \geq \omega_1$  be a regular cardinal. According to Lemma 6.14 and Theorem 6.11  $\text{HCK}(\text{CH}(\mathbf{A} \cup \{C(\alpha)_n^-\})) = \mathbf{FG}$  for each  $n$ ,  $0 < n < \omega_0$ . Thus, we get  $C(\alpha)_n^- \in \mathbf{A}$  for each  $n$  and by Proposition 6.4  $\text{SCH}(C(\alpha)) \subseteq \mathbf{A}$ , a contradiction.

The proof that  $\mathcal{L}$  has no maximal elements is analogous.  $\square$

The family  $\text{CH}(\mathbf{A}_0 \cup \{C(\alpha)_n^-\})$ ,  $0 < n < \omega_0$ , where  $\alpha \geq \omega_1$  is a regular cardinal, is an example of a countable family of elements of  $\mathcal{L}$  such that its join does not belong to  $\mathcal{L}$ .

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DEPARTMENT OF ALGEBRA AND NUMBER THEORY, FMFI UK, MLYNSKÁ DOLINA, 842 48 BRATISLAVA, SLOVAKIA

*E-mail address:* sleziak@fmph.uniba.sk